Indian Statistical Institute
First Semester Back Paper Exam, 2006-2007
M.Math.II Year
Operator Theory

Time: 3 hrs Date: 05-01-07 Marks : $10 \times 10 = 100$

Instructor: T S S R K Rao

1. Let X be a Banach space and M, N are closed subspaces such that $M \cap N = \{0\}$ and $X = M + N$. Show that $P : X \to M$ defined by $p(m+n) = m$ is a continuous, linear projection.

2. Let $T : L^2[0,1] \to L^2[0,1]$ be defined by $(Tf)(x) = i \int\limits_0^1 K(x,y)\, f(y)\, dy$ where

$$\begin{aligned} K(x,y) &= 1 \text{ if } y \leq x \\ &= -1 \text{ otherwise.} \end{aligned}$$

(Here $f \in L^2[0,1], x, y \in [0,1]$.) Find the eigenvalues of T.

3. Let X be a Banach space and let $\{T_n\}_{n \geq 1}$ be a sequence of compact operators such that $T_n(x) \to (x)\ \forall\ x \in X$. Show that $\{T_n\}_{n \geq 1}$ in a one-sided approximate identity of the Banach algebra $\mathcal{K}(X)$.

4. Let X be a Banach space and let $T : X \to X$ be a bounded linear operators. If the range of T^* is closed. Show that range of T is closed.

5. Show that $\mathcal{K}(\ell^2)$ is a separable Banach space.

6. Let A be a C^*-algebra of compact operators such that A has no non-trivial ideals. Show that A is isometric to $\mathcal{K}(H^1)$ for some Hilbert space H^1.

7. Show that any C^*-algebra without identity can be embedded as an ideal in a C^*-algebra with identity.

8. Let f be a state on a C^*-algebra A with identity. Show that there is a representation π of A on a Hilbert space H and a $\xi \in H$ such that $f(x) = \langle \pi(x)\xi, \xi \rangle$.

9. Show that the set of positive elements of a C^*-algebra with identity is a convex set.

10. Let $\hat{A}$ be the C^*-algebra obtained by adjoining the identity e, to the C^*-algebra A. Let $f : A \to \mathbb{C}$ be a bounded positive linear map. Show that $\hat{f} : \tilde{A} \to \mathbb{C}$ defined by $\hat{f}(x + \lambda e) = f(x) + \lambda \|f\|$ is a positive linear functional.